

Curvature of a class of indefinite globally framed f -manifolds

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Abstract

We present a compared analysis of some properties of indefinite almost \mathcal{S} -manifolds and indefinite \mathcal{S} -manifolds. We give some characterizations in terms of the Levi-Civita connection and of the characteristic vector fields. We study the sectional and φ -sectional curvature of indefinite almost \mathcal{S} -manifolds and state an expression of the curvature tensor field for the indefinite \mathcal{S} -space forms. We analyse the sectional curvature of indefinite \mathcal{S} -manifold in which the number of the spacelike characteristic vector fields is equal to that of the timelike characteristic vector fields. Some examples are also described.

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1 Introduction

In the framework of Riemannian geometry, almost \mathcal{S} -manifolds and \mathcal{S} -manifolds represent a natural generalization of contact and Sasakian manifolds, respectively. Such manifolds have been extensively studied by several authors and from different points of view ([2, 3, 4, 7, 8, 12]). On the other hand, also Sasakian manifolds with semi-Riemannian metric have been considered ([10, 6, 17]), and in recent works many authors, (for example, in [13], K.L. Duggal and B. Sahin) study lightlike submanifolds of indefinite Sasakian manifolds. Indefinite \mathcal{S} -manifolds are natural generalizations of indefinite Sasakian manifolds. Moreover many spacetime manifolds can be endowed with f -structures ([9]).

After a first section on f -structures and indefinite metric $g.f.f$ -structures, in section 3, we carry out an in-depth study of the indefinite (almost) \mathcal{S} -manifolds. In section 4 we describe two examples of 6-dimensional indefinite \mathcal{S} -manifolds having two characteristic vector fields which are both spacelike or both timelike. A third example is a Lorentzian indefinite \mathcal{S} -manifold of dimension 4 with two characteristic vector fields of different causal type. In section 5, after some Lemmas, we prove that the φ -sectional curvatures completely determine the sectional curvatures. Then, we find an expression of the curvature tensor field R which characterizes the indefinite \mathcal{S} -space forms, that is indefinite \mathcal{S} -manifolds with constant φ -sectional curvature. Then, in section 6, we consider the curvature of special indefinite \mathcal{S} -manifold in which the number of the characteristic vector fields is even with an equal number of spacelike and timelike characteristic vector fields; we prove that the special indefinite \mathcal{S} -manifold described in the third example in section 4 turns out to be an indefinite \mathcal{S} -space form whose φ -sectional curvature vanishes.

All manifolds and tensor fields are assumed to be smooth.

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2 Indefinite metric f -structure

We recall that an f -structure on a manifold M is a non null $(1, 1)$ -tensor field φ on M of constant rank such that $\varphi^3 + \varphi = 0$. A manifold M , provided with an f -structure, is said to be an f -manifold, and it is known that TM splits into two complementary subbundles $\text{Im } \varphi$ and $\ker \varphi$ and that the restriction of φ to $\text{Im } \varphi$ determines a complex structure on it and the rank of φ is even. An interesting case of f -structure occurs when $\ker \varphi$ is parallelizable for which there exist global vector fields ξ_α , $\alpha \in \{1, \dots, r\}$, with their dual 1-forms η^α , satisfying: $\varphi^2 = -I + \sum_{\alpha=1}^r \eta^\alpha \otimes \xi_\alpha$, and $\eta^\alpha(\xi_\beta) = \delta_\beta^\alpha$. Such an f -structure is called an f -structure with parallelizable kernel or globally framed f -structure, briefly denoted $g.f.f$ -structure ([14]). Moreover, a manifold M endowed with a $g.f.f$ -structure is called a $g.f.f$ -manifold, and it is denoted with $(M, \varphi, \xi_\alpha, \eta^\alpha)$; the vector fields ξ_α , ($\alpha = 1, \dots, r$), are called *characteristic vector fields*.

It is also known that an f -structure, on a manifold M , is called *normal* if the tensor field $N = N_\varphi + 2 \sum_{\alpha=1}^r d\eta^\alpha \otimes \xi_\alpha$ vanishes, where N_φ is the Nijenhuis torsion of φ .

Definition 2.1 Let (M, φ) be a $(2n+r)$ -dimensional f -manifold and g a semi-Riemannian metric on M with index ν , $0 < \nu < 2n+r$. Then, the pair (φ, g) is said to be an *indefinite metric f -structure*, and the triple (M, φ, g) is called an *indefinite metric f -manifold*, if φ is skew-symmetric with respect to g , that is, for any $X, Y \in \Gamma(TM)$:

$$g(\varphi X, Y) + g(X, \varphi Y) = 0.$$

Definition 2.2 Let $(M^{2n+r}, \varphi, \xi_\alpha, \eta^\alpha)$ be a $g.f.f$ -manifold, and g a semi-Riemannian metric on M with index ν , $0 < \nu < 2n+r$. Then, we say that the two structures are *compatible* if for any $X, Y \in \Gamma(TM)$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^r \varepsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y), \quad \varepsilon_\alpha g(X, \xi_\alpha) = \eta^\alpha(X) \quad \text{for any } \alpha \in \{1, \dots, r\}, \quad (1)$$

where $\varepsilon_\alpha = \pm 1$ according to whether ξ_α is spacelike or timelike. Then $(M^{2n+r}, \varphi, \xi_\alpha, \eta^\alpha, g)$ is called an *indefinite metric $g.f.f$ -manifold*.

We shall use the Einstein convention omitting the sum symbol for repeated indices above and below, writing, e.g., $\varepsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y)$ to mean $\sum_{\alpha=1}^r \varepsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y)$.

Observe that if g is a semi-Riemannian metric on a $g.f.f$ -manifold $(M, \varphi, \xi_\alpha, \eta^\alpha)$ compatible with the f -structure φ , then the pair (φ, g) is necessarily an indefinite metric f -structure. The fundamental 2-form Φ is defined putting $\Phi(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(TM)$. Let $(M, \varphi, \xi_\alpha, \eta^\alpha)$, with $\alpha = 1, \dots, r$, be a $g.f.f$ -manifold, and g a compatible semi-Riemannian metric on M . We know that the orthogonal decomposition $TM = \text{Im } \varphi \oplus \ker \varphi$ holds, and that the induced structure J on $\text{Im } \varphi$ is an almost complex structure; then $(\text{Im } \varphi, g|_{\text{Im } \varphi}, J)$ is a indefinite Hermitian distribution and the only possible signatures of g are $(2p, 2q)$ with $p+q = n$; therefore g cannot be a Lorentz metric, for $n > 1$. We shall denote $\text{Im } \varphi$ and $\ker \varphi$ with \mathfrak{D} and \mathfrak{D}^\perp respectively and for a section of \mathfrak{D} (\mathfrak{D}^\perp) we will write $X \in \mathfrak{D}$ or $X \in \Gamma(\mathfrak{D})$ ($X \in \mathfrak{D}^\perp$ or $X \in \Gamma(\mathfrak{D}^\perp)$).

We recall the following result due to A. Bejancu and K.L. Duggal ([10]).

Theorem 2.3 Let $(M, \varphi, \xi_\alpha, \eta^\alpha)$, $\alpha = 1, \dots, r$, be a $g.f.f$ -manifold and h_0 a semi-Riemannian metric on M ; we suppose that $\{\xi_\alpha\}_{1 \leq \alpha \leq r}$ are h_0 -orthonormal and that $h_0(\xi_\alpha, \xi_\alpha) = -\varepsilon_\alpha$, for any $\alpha \in \{1, \dots, r\}$. Then there exists a symmetric tensor field g of type $(0, 2)$ on M satisfying (1).

Now, with a standard computation as in the Riemannian setting ([2]), one can prove the following results.

Proposition 2.4 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric g.f.f-manifold. Then, the Levi-Civita connection satisfies the following equality, for any $X, Y, Z \in \Gamma(TM)$:*

$$2g((\nabla_X \varphi)Y, Z) = 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) + g(N(Y, Z), \varphi X) + \varepsilon_\alpha N_\alpha^{(2)}(Y, Z)\eta^\alpha(X) + 2\varepsilon_\alpha d\eta^\alpha(\varphi Y, X)\eta^\alpha(Z) - 2\varepsilon_\alpha d\eta^\alpha(\varphi Z, X)\eta^\alpha(Y), \quad (2)$$

where $N_\alpha^{(2)}(X, Y) = (\mathcal{L}_{\varphi X} \eta^\alpha)(Y) - (\mathcal{L}_{\varphi Y} \eta^\alpha)(X) = 2d\eta^\alpha(\varphi X, Y) - 2d\eta^\alpha(\varphi Y, X)$.

Proposition 2.5 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric g.f.f-manifold. Then the following statements hold:*

- a) $(\mathcal{L}_{\xi_\alpha} \Phi)(X, Y) = (\mathcal{L}_{\xi_\alpha} g)(X, \varphi Y) + g(X, (\mathcal{L}_{\xi_\alpha} \varphi)Y)$, for any $\alpha \in \{1, \dots, r\}$.
- b) $(\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \varphi)Z)$, for any $X, Y, Z \in \Gamma(TM)$.
- c) If $\mathcal{L}_{\xi_\alpha} \varphi = 0$, then $\eta^\beta[\varphi Z, \xi_\alpha] = 0$, for any $\beta \in \{1, \dots, r\}$.
- d) $N = 0 \Rightarrow N_\alpha^{(2)} = 0$, for any $\alpha \in \{1, \dots, r\}$.

Between the indefinite metric g.f.f-manifolds, we can define the following classes.

Definition 2.6 Let $(M^{2n+r}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric g.f.f-manifold. M is called *indefinite \mathcal{K} -manifold* if it is normal and $d\Phi = 0$.

In this case $\mathcal{L}_{\xi_\alpha} \Phi = i_{\xi_\alpha} d\Phi + di_{\xi_\alpha} \Phi = 0$, therefore, from a) of Proposition 2.5, we obtain that $\mathcal{L}_{\xi_\alpha} \varphi = 0$ if and only if the characteristic vector fields ξ_α are Killing. Two subclasses of indefinite \mathcal{K} -manifolds are those of indefinite \mathcal{C} -manifolds and indefinite \mathcal{S} -manifolds, that are defined as follows: an indefinite \mathcal{K} -manifold is called *indefinite \mathcal{C} -manifold* if $d\eta^\alpha = 0$ for any $\alpha \in \{1, \dots, r\}$, while it is called *indefinite \mathcal{S} -manifold* if $d\eta^\alpha = \Phi$ for any $\alpha \in \{1, \dots, r\}$.

3 Indefinite \mathcal{S} -manifolds

The properties of (almost) \mathcal{S} -manifolds (with Riemannian metric) are studied in [12] and in [2]. Now, we discuss indefinite (almost) \mathcal{S} -manifolds and their properties.

3.1 Indefinite almost \mathcal{S} -manifolds

Definition 3.1 Let $(M^{2n+r}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric g.f.f-manifold. M is called *indefinite almost \mathcal{S} -manifold* if $d\eta^\alpha = \Phi$ for any $\alpha \in \{1, \dots, r\}$.

Lemma 3.2 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost \mathcal{S} -manifold. Then the tensor fields $N_\alpha^{(2)}$ vanish and for any $X, Y \in \Gamma(\mathfrak{D})$ and $\alpha \in \{1, \dots, r\}$, we have*

$$\eta^\alpha[\varphi X, Y] = \eta^\alpha[\varphi Y, X]$$

Proof. For $\alpha \in \{1, \dots, r\}$, we have $N_\alpha^{(2)}(X, Y) = 2d\eta^\alpha(\varphi X, Y) - 2d\eta^\alpha(\varphi Y, X) = 2\Phi(\varphi X, Y) - 2\Phi(\varphi Y, X) = 0$. Then, for any $X, Y \in \Gamma(\mathfrak{D})$, $2d\eta^\alpha(\varphi X, Y) = -\eta^\alpha([\varphi X, Y])$ implies $\eta^\alpha[\varphi X, Y] = \eta^\alpha[\varphi Y, X]$. \square

Proposition 3.3 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost \mathcal{S} -manifold and $\bar{\eta} := \sum_{\alpha=1}^r \varepsilon_\alpha \eta^\alpha$. Then, the following statements hold:*

$$2g((\nabla_X \varphi)Y, Z) = g(N(Y, Z), \varphi X) + 2g(\varphi Y, \varphi X)\bar{\eta}(Z) - 2g(\varphi Z, \varphi X)\bar{\eta}(Y), \quad (3)$$

$$\nabla_{\xi_\alpha} \varphi = 0 \quad \text{for all } \alpha \in \{1, \dots, r\}, \quad \nabla_{\xi_\alpha} \xi_\beta = 0 \quad \text{for all } \alpha, \beta \in \{1, \dots, r\}. \quad (4)$$

Proof. Equation (3) follows from (2) using $d\Phi = 0$, $N_\alpha^{(2)} = 0$ and $d\eta^\alpha = \Phi$, for $\alpha \in \{1, \dots, r\}$. Then, putting $X = \xi_\alpha$, we obtain $\nabla_{\xi_\alpha} \varphi = 0$.

Hence, we have $0 = (\nabla_{\xi_\alpha} \varphi)(\xi_\beta) = -\varphi(\nabla_{\xi_\alpha} \xi_\beta)$, therefore $\nabla_{\xi_\alpha} \xi_\beta \in \mathfrak{D}^\perp$, which implies that $[\xi_\alpha, \xi_\beta] \in \mathfrak{D}^\perp$. On the other hand, for any $\gamma \in \{1, \dots, r\}$

$$0 = \Phi(\xi_\alpha, \xi_\beta) = d\eta^\gamma(\xi_\alpha, \xi_\beta) = -\frac{1}{2}\eta^\gamma[\xi_\alpha, \xi_\beta] = -\frac{1}{2}\varepsilon_\gamma g([\xi_\alpha, \xi_\beta], \xi_\gamma).$$

Therefore $[\xi_\alpha, \xi_\beta] \in \mathfrak{D} \cap \mathfrak{D}^\perp$ and we obtain $[\xi_\alpha, \xi_\beta] = 0$ and $\nabla_{\xi_\alpha} \xi_\beta = \nabla_{\xi_\beta} \xi_\alpha$. Now we check that $\nabla_{\xi_\alpha} \xi_\beta \in \mathfrak{D}$, that is, for any $\gamma \in \{1, \dots, r\}$, $g(\nabla_{\xi_\alpha} \xi_\beta, \xi_\gamma) = 0$. Being $g(\xi_\beta, \xi_\gamma) = \varepsilon_\beta \delta_{\beta\gamma}$ and using the covariant derivative with respect to ξ_α , we find $g(\nabla_{\xi_\alpha} \xi_\beta, \xi_\gamma) + g(\xi_\beta, \nabla_{\xi_\alpha} \xi_\gamma) = 0$, and, covariantly differentiating $g(\xi_\alpha, \xi_\gamma) = \varepsilon_\alpha \delta_{\alpha\gamma}$ with respect to ξ_β , we obtain $g(\nabla_{\xi_\beta} \xi_\alpha, \xi_\gamma) + g(\xi_\alpha, \nabla_{\xi_\beta} \xi_\gamma) = 0$. From the last two equations, using $\nabla_{\xi_\alpha} \xi_\beta = \nabla_{\xi_\beta} \xi_\alpha$, we have $g(\xi_\beta, \nabla_{\xi_\alpha} \xi_\gamma) = g(\xi_\alpha, \nabla_{\xi_\beta} \xi_\gamma)$. Therefore, $g(\nabla_{\xi_\alpha} \xi_\beta, \xi_\gamma) = g(\xi_\alpha, \nabla_{\xi_\gamma} \xi_\beta) = g(\xi_\alpha, \nabla_{\xi_\beta} \xi_\gamma) = -g(\nabla_{\xi_\beta} \xi_\alpha, \xi_\gamma) = -g(\nabla_{\xi_\alpha} \xi_\beta, \xi_\gamma)$, from which $g(\nabla_{\xi_\alpha} \xi_\beta, \xi_\gamma) = 0$ follows. This result and $\nabla_{\xi_\alpha} \xi_\beta \in \mathfrak{D}^\perp$ imply $\nabla_{\xi_\alpha} \xi_\beta = 0$. \square

Proposition 3.4 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost \mathcal{S} -manifold. Then*

- a) *for any $\alpha \in \{1, \dots, r\}$ the operator $h_\alpha = \frac{1}{2}\mathcal{L}_{\xi_\alpha} \varphi$ is self-adjoint,*
- b) *for any $\alpha, \beta \in \{1, \dots, r\}$, $h_\alpha(\xi_\beta) = 0$,*
- c) *for any $\alpha \in \{1, \dots, r\}$, $h_\alpha \circ \varphi + \varphi \circ h_\alpha = 0$.*

Proof. As first step, using (4), for any $X, Y \in \Gamma(TM)$ and any $\alpha \in \{1, \dots, r\}$, we easily obtain,

$$g((\mathcal{L}_{\xi_\alpha} \varphi)X, Y) = \varepsilon_\alpha(-(\varphi X)(\eta^\alpha(Y)) + \eta^\alpha(\nabla_{\varphi X} Y + \nabla_X(\varphi Y))).$$

It follows that

$$\begin{aligned} 2g(h_\alpha(X), Y) - 2g(h_\alpha(Y), X) &= -\varepsilon_\alpha(\varphi X)(\eta^\alpha(Y)) + \varepsilon_\alpha \eta^\alpha[\varphi X, Y] + \varepsilon_\alpha(\varphi Y)(\eta^\alpha(X)) \\ &\quad - \varepsilon_\alpha \eta^\alpha[\varphi Y, X] = -\varepsilon_\alpha(\mathcal{L}_{\varphi X} \eta^\alpha)(Y) + \varepsilon_\alpha(\mathcal{L}_{\varphi Y} \eta^\alpha)(X) = 0. \end{aligned}$$

Obviously, for any $\alpha, \beta \in \{1, \dots, r\}$ we have $h_\alpha(\xi_\beta) = 0$ and finally

$$\begin{aligned} 2(h_\alpha \circ \varphi + \varphi \circ h_\alpha)(X) &= \mathcal{L}_{\xi_\alpha}(\varphi^2 X) - \varphi(\mathcal{L}_{\xi_\alpha}(\varphi X)) + \varphi(\mathcal{L}_{\xi_\alpha}(\varphi X)) - \varphi(\mathcal{L}_{\xi_\alpha} X) \\ &= \xi_\alpha(\eta^\beta(X))\xi_\beta - \eta^\beta[\xi_\alpha, X]\xi_\beta = 0 \end{aligned}$$

for any $\alpha \in \{1, \dots, r\}$ and any $X \in \Gamma(TM)$. \square

Proposition 3.5 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost \mathcal{S} -manifold. Then, for any $X, Y \in \Gamma(TM)$, the following properties hold:*

- a) $\varphi(N(X, Y)) + N(\varphi X, Y) = 2\eta^\alpha(X)h_\alpha(Y)$,
- b) $N(X, Y) \in \mathfrak{D}$.

Proof. Using Lemma 3.2, we obtain

$$\begin{aligned} \varphi(N(X, Y)) + N(\varphi X, Y) &= -(\mathcal{L}_{\varphi Y} \eta^\alpha)(X)\xi_\alpha + (\mathcal{L}_{\varphi X} \eta^\alpha)(Y)\xi_\alpha + \eta^\alpha(X)(\mathcal{L}_{\xi_\alpha} \varphi)(Y) \\ &= 2\eta^\alpha(X)h_\alpha(Y). \end{aligned}$$

Now, we observe that for any $\alpha \in \{1, \dots, r\}$ we have $[\xi_\alpha, \mathfrak{D}] \subset \mathfrak{D}$, in fact, if $\beta \in \{1, \dots, r\}$ and $X \in \Gamma(TM)$, we have $\eta^\beta[\xi_\alpha, \varphi X] = -2d\eta^\beta(\xi_\alpha, \varphi X) = 0$ and in particular, if $X \in \mathfrak{D}$ and $\alpha = \beta$,

we get $\eta^\alpha[\xi_\alpha, X] = 0$. So, if $Z \in \mathfrak{D}$ then $N(\xi_\alpha, Z) = -[\xi_\alpha, Z] - \varphi[\xi_\alpha, \varphi Z] \in \mathfrak{D}$. It is easy to check that $N(\xi_\alpha, \xi_\beta) = 0$ for any $\alpha, \beta \in \{1, \dots, r\}$; therefore, we have that $N(\xi_\alpha, X) \in \mathfrak{D}$ for any $X \in \Gamma(TM)$. Finally, applying a), we have $g(N(\varphi X, Y), \xi_\alpha) = 2\eta^\beta(X)g(h_\beta(Y), \xi_\alpha) = 0$. Hence, if $X, Y \in \Gamma(TM)$, we get $N(X, Y) = -N(\varphi^2 X, Y) + \eta^\alpha(X)N(\xi_\alpha, Y)$, and being $N(\varphi^2 X, Y) \in \mathfrak{D}$ and $N(\xi_\alpha, Y) \in \mathfrak{D}$, we conclude that $N(X, Y) \in \mathfrak{D}$. \square

Proposition 3.6 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost \mathcal{S} -manifold. For any $X \in \Gamma(TM)$ and for any $\alpha \in \{1, \dots, r\}$,*

$$\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi(X) - \varphi(h_\alpha X).$$

Proof. Putting $X = \xi_\alpha$ in a) of Proposition 3.5, we have that for any $Z, Y \in \Gamma(TM)$

$$g(N(\xi_\alpha, Y), \varphi Z) = -g(\varphi(N(\xi_\alpha, Y)), Z) = -2\eta^\beta(\xi_\alpha)g(h_\beta(Y), Z) = -2g(h_\alpha(Y), Z).$$

Moreover, applying (3) of Proposition 3.3, for any $\alpha \in \{1, \dots, r\}$ we find:

$$\begin{aligned} g(-\varphi(\nabla_X \xi_\alpha), Z) &= \frac{1}{2}g(N(\xi_\alpha, Z), \varphi X) - g(\varphi Z, \varphi X)\eta(\xi_\alpha) \\ &= -g(h_\alpha(Z), X) - \varepsilon_\alpha g(Z, X) + \varepsilon_\alpha \varepsilon_\beta \eta^\beta(X)\eta^\beta(Z) \\ &= g(-h_\alpha(X) - \varepsilon_\alpha X + \varepsilon_\alpha \eta^\beta(X)\xi_\beta, Z), \end{aligned}$$

then $\varphi(\nabla_X \xi_\alpha) = h_\alpha(X) + \varepsilon_\alpha X - \varepsilon_\alpha \eta^\beta(X)\xi_\beta$, and, applying φ , we complete the proof. Note that $\nabla_X \xi_\alpha \in \mathfrak{D}$. \square

Proposition 3.7 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost \mathcal{S} -manifold. For $X, Y \in \Gamma(TM)$, we have*

$$(\nabla_X \varphi)(Y) + (\nabla_{\varphi X} \varphi)(\varphi Y) = 2g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X) - \eta^\alpha(Y)h_\alpha(X).$$

where $\bar{\xi} := \sum_{\alpha=1}^r \xi_\alpha$ and $\bar{\eta}(X) = g(X, \bar{\xi})$, for any $X \in \Gamma(TM)$.

Proof. Using (3), Proposition 3.5 and Proposition 3.6, for any $X, Y, Z \in \Gamma(TM)$ we have

$$\begin{aligned} 2g((\nabla_X \varphi)(Y), Z) + 2g((\nabla_{\varphi X} \varphi)(\varphi Y), Z) &= -g(\varphi(N(Y, Z)) + N(\varphi Y, Z), X) \\ &\quad + 4g(\varphi Y, \varphi X)\bar{\eta}(Z) - 2g(\varphi Z, \varphi X)\bar{\eta}(Y) \\ &= -2g(Z, \eta^\alpha(Y)h_\alpha(X)) + 4g(\varphi Y, \varphi X)g(Z, \bar{\xi}) \\ &\quad + 2g(Z, \bar{\eta}(Y)\varphi^2 X). \end{aligned}$$

Then, we deduce $(\nabla_X \varphi)(Y) + (\nabla_{\varphi X} \varphi)(\varphi Y) = 2g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X) - \eta^\alpha(Y)h_\alpha(X)$. Obviously, $\bar{\eta}(X) = \sum_{\alpha=1}^r \varepsilon_\alpha \eta^\alpha(X) = \sum_{\alpha=1}^r g(X, \xi_\alpha) = g(X, \bar{\xi})$. \square

Corollary 3.8 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost \mathcal{S} -manifold. Then, for any $X, Y \in \mathfrak{D}$:*

- a) $(\nabla_X \varphi)(Y) + (\nabla_{\varphi X} \varphi)(\varphi Y) = 2g(X, Y)\bar{\xi}$,
- b) $(\nabla_X \varphi)(\varphi X) = (\nabla_{\varphi X} \varphi)(X)$.

Proof. The first statement follows from the above proposition. Putting $Y := \varphi X$ in a), we have $(\nabla_X \varphi)(\varphi X) + (\nabla_{\varphi X} \varphi)(\varphi^2 X) = 2g(X, \varphi X)\bar{\xi} = 0$, therefore, being $\varphi^2 X = -X$, we obtain $(\nabla_X \varphi)(\varphi X) = (\nabla_{\varphi X} \varphi)(X)$. \square

Remark 3.9 The statement b) can be written as $\nabla_X(\varphi^2 X) - \varphi(\nabla_X \varphi X) = \nabla_{\varphi X}(\varphi X) - \varphi(\nabla_{\varphi X} X)$, i.e. as $\nabla_X X + \nabla_{\varphi X}(\varphi X) = \varphi[\varphi X, X]$.

3.2 Indefinite \mathcal{S} -manifolds

Definition 3.10 Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric $g.f.f$ -manifold. M is said an *indefinite \mathcal{S} -manifold* if it is a normal indefinite almost \mathcal{S} -manifold.

Proposition 3.11 Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost \mathcal{S} -manifold. Then M is an indefinite \mathcal{S} -manifold if and only if, for any $X, Y \in \Gamma(TM)$, the Levi-Civita connection satisfies:

$$(\nabla_X \varphi)Y = g(X, Y)\bar{\xi} - \bar{\eta}(Y)X - \varepsilon_\alpha \eta^\alpha(X)\eta^\alpha(Y)\bar{\xi} + \bar{\eta}(Y)\eta^\alpha(X)\xi_\alpha,$$

or equivalently

$$(\nabla_X \varphi)Y = g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X). \quad (5)$$

Proof. Assuming that M is an indefinite \mathcal{S} -manifold, (3) becomes

$$g((\nabla_X \varphi)Y, Z) = g(\varphi Y, \varphi X)\bar{\eta}(Z) - g(\varphi Z, \varphi X)\bar{\eta}(Y) = g(Z, g(\varphi Y, \varphi X)\bar{\xi} + \bar{\eta}(Y)\varphi^2 X),$$

from which

$$(\nabla_X \varphi)Y = g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X) = g(X, Y)\bar{\xi} - \varepsilon_\alpha \eta^\alpha(X)\eta^\alpha(Y)\bar{\xi} - \bar{\eta}(Y)X + \bar{\eta}(Y)\eta^\alpha(X)\xi_\alpha.$$

Vice versa, we suppose that ∇ satisfies (5). Then we obtain $g((\nabla_X \varphi)Y, Z) = g(\varphi Y, \varphi X)\bar{\eta}(Z) - g(\varphi Z, \varphi X)\bar{\eta}(Y)$, and comparing with (3), we deduce for any $X, Y \in \Gamma(TM)$, $g(N(Y, Z), \varphi X) = 0$. From Proposition 3.5, we obtain that $N(Y, Z) = 0$ for any $Y, Z \in \Gamma(TM)$, that is M is normal. \square

Remark 3.12 In an indefinite \mathcal{S} -manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, the operators $\mathcal{L}_{\xi_\alpha} \varphi$, and then h_α , vanish. In fact, by direct computation for any $X \in \Gamma(TM)$ and for any $\alpha \in \{1, \dots, r\}$ we get $N(\varphi X, \xi_\alpha) = (\mathcal{L}_{\xi_\alpha} \varphi)X = 2h_\alpha(X)$, and the normality condition implies $h_\alpha = 0$. Using Proposition 3.6, we obtain, for any $\alpha \in \{1, \dots, r\}$, $\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi X$.

Now, we give the condition of indefinite \mathcal{S} -manifold in terms of the fundamental 2-form:

Proposition 3.13 Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost \mathcal{S} -manifold. Then M is an indefinite \mathcal{S} -manifold if and only if for any $X, Y, Z \in \Gamma(TM)$:

$$(\nabla_X \Phi)(Y, Z) = \bar{\eta}(Y)g(\varphi X, \varphi Z) - \bar{\eta}(Z)g(\varphi X, \varphi Y). \quad (6)$$

Proof. One simply uses $(\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \varphi)Z)$ in (5). \square

Proposition 3.14 Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric $g.f.f$ -manifold. If the vector fields ξ_α are Killing, $\mathcal{L}_{\xi_\alpha} \eta^\beta = 0$ for any $\alpha, \beta \in \{1, \dots, r\}$ and M satisfies (5) or equivalently (6), then M is an indefinite \mathcal{S} -manifold.

Proof. Being $3d\Phi(X, Y, Z) = \mathfrak{S}_{X, Y, Z}(\nabla_X \Phi)(Y, Z)$, from (6) we get $d\Phi = 0$ and $(\mathcal{L}_{\xi_\alpha} \Phi)(X, Y) = 0$, since $\mathcal{L}_{\xi_\alpha} \Phi = i_{\xi_\alpha} d\Phi + di_{\xi_\alpha} \Phi$. Proposition 2.5 implies $(\mathcal{L}_{\xi_\alpha} g)(X, \varphi Y) + g(X, (\mathcal{L}_{\xi_\alpha} \varphi)Y) = 0$, for any $\alpha \in \{1, \dots, r\}$ and $X, Y \in \Gamma(TM)$. Hence, being ξ_α a Killing vector field, we find $\mathcal{L}_{\xi_\alpha} \varphi = 0$ and then $\eta^\beta([\xi_\alpha, \varphi Y]) = 0$, for any $\alpha, \beta \in \{1, \dots, r\}$. In these hypotheses, (2) becomes

$$\begin{aligned} 2g((\nabla_X \varphi)Y, Z) &= g(N(Y, Z), \varphi X) + 2\varepsilon_\alpha [d\eta^\alpha(\varphi Y, Z)\eta^\alpha(X) - d\eta^\alpha(\varphi Z, Y)\eta^\alpha(X) \\ &\quad + d\eta^\alpha(\varphi Y, X)\eta^\alpha(Z) - d\eta^\alpha(\varphi Z, X)\eta^\alpha(Y)]. \end{aligned}$$

On the other hand, (6) implies $g(Y, (\nabla_X \varphi)Z) = \bar{\eta}(Y)g(\varphi X, \varphi Z) - \bar{\eta}(Z)g(\varphi X, \varphi Y)$, therefore we deduce

$$\begin{aligned} g(N(Y, Z), \varphi X) &= -2\varepsilon_\alpha [(d\eta^\alpha(\varphi Y, Z) - d\eta^\alpha(\varphi Z, Y))\eta^\alpha(X) + (d\eta^\alpha(\varphi Y, X) - g(\varphi X, \varphi Y))\eta^\alpha(Z) \\ &\quad - (d\eta^\alpha(\varphi Z, X) - g(\varphi X, \varphi Z))\eta^\alpha(Y)]. \end{aligned}$$

Putting $Y = \xi_\beta$ in the above equation, we get

$$g(N(\xi_\beta, Z), \varphi X) = 2\varepsilon_\beta(d\eta^\beta(\varphi Z, X) - g(\varphi X, \varphi Z)). \quad (7)$$

Since $N(\xi_\beta, Z) = -[\xi_\beta, Z] - \varphi[\xi_\beta, \varphi Z] + \xi_\beta(\eta^\alpha(Z))\xi_\alpha$, then $\varphi N(\xi_\beta, Z) = (\mathcal{L}_{\xi_\alpha}\varphi)Z - \eta^\alpha[\xi_\beta, \varphi Z]\xi_\alpha = 0$ and (7) gives $d\eta^\beta(\varphi Z, X) = g(\varphi X, \varphi Z) = \Phi(\varphi Z, X)$. Finally, $\mathcal{L}_{\xi_\alpha}\bar{\eta}^\beta = 0$ implying $i_{\xi_\alpha}d\bar{\eta}^\beta = 0$ and being $Y = -\varphi^2Y + \eta^\alpha(Y)\xi_\alpha$, for any $Y \in \Gamma(TM)$, we obtain $d\eta^\beta(Y, X) = -d\eta^\beta(\varphi^2Y, X) + \eta^\alpha(Y)d\eta^\beta(\xi_\alpha, X) = -\Phi(\varphi^2Y, X) = \Phi(Y, X)$. Then M is an indefinite almost \mathcal{S} -manifold and we apply Proposition 3.11. \square

4 Examples of indefinite \mathcal{S} -manifolds

We describe some examples of indefinite \mathcal{S} -manifolds, where the characteristic vector fields are either timelike or spacelike or of both types.

Example 4.1 We consider \mathbb{R}^6 with its standard coordinates $\{x^1, x^2, y^1, y^2, z^1, z^2\}$. We introduce on \mathbb{R}^6 an indefinite $g.f.f$ -structure $(\varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ by setting

$$\xi_\alpha = \frac{\partial}{\partial z^\alpha}, \quad \eta^\alpha = dz^\alpha - \sum_{i=1}^2 y^i dx^i, \quad \alpha \in \{1, 2\},$$

$$g = -\sum_{\alpha=1}^2 \eta^\alpha \otimes \eta^\alpha + \frac{1}{2} \sum_{i=1}^2 ((dx^i)^2 + (dy^i)^2),$$

and φ given, with respect to the frame $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \xi_1, \xi_2\}$, by the matrix

$$F = \begin{pmatrix} 0 & I_2 & 0 \\ -I_2 & 0 & 0 \\ 0 & Y & 0 \end{pmatrix}, \quad \text{where } Y = \begin{pmatrix} y^1 & y^2 \\ y^1 & y^2 \end{pmatrix}.$$

We put $M = (\mathbb{R}_2^6, \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$. A straightforward computation shows that g is a metric tensor field. Firstly we check that g is non-degenerate and then we compute its index. The matrix G of g is given by

$$G = \begin{pmatrix} \frac{1}{2} - 2(y^1)^2 & -2y^1y^2 & 0 & 0 & y^1 & y^1 \\ -2y^1y^2 & \frac{1}{2} - 2(y^2)^2 & 0 & 0 & y^2 & y^2 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ y^1 & y^2 & 0 & 0 & -1 & 0 \\ y^1 & y^2 & 0 & 0 & 0 & -1 \end{pmatrix},$$

and $\det G = \frac{1}{16} \neq 0$. Now, to determine the index of g , we look for the eigenvalues of G . Since

$$\det(G - \lambda I) = -(\frac{1}{2} - \lambda)^3(1 + \lambda)(\lambda^2 + (2(y^1)^2 + 2(y^2)^2 + \frac{1}{2})\lambda - \frac{1}{2}),$$

we find that the index of g is two; therefore g is a semi-Riemannian metric of the index 2 on \mathbb{R}^6 . We remark that ξ_1 and ξ_2 are timelike vector fields. It is easy to prove that M is an indefinite \mathcal{S} -manifold.

Example 4.2 The second example of an indefinite \mathcal{S} -manifold is $M = (\mathbb{R}_2^6, \varphi, \xi_\alpha, \eta^\alpha, g)$, where, for any $\alpha \in \{1, 2\}$, we put

$$\xi_\alpha := \frac{\partial}{\partial z^\alpha}, \quad \eta^\alpha := dz^\alpha - \sum_{i=1}^2 \tau_i y^i dx^i,$$

φ, g are given by

$$F = \begin{pmatrix} 0 & I_2 & 0 \\ -I_2 & 0 & 0 \\ 0 & Y & 0 \end{pmatrix}, \quad \text{where } Y = \begin{pmatrix} -y^1 & y^2 \\ -y^1 & y^2 \end{pmatrix},$$

and

$$g = \sum_{\alpha=1}^2 \eta^\alpha \otimes \eta^\alpha + \frac{1}{2} \sum_{i=1}^2 \tau_i ((dx^i)^2 + (dy^i)^2),$$

respectively, where $\tau_i = \mp 1$ according to whether $i = 1$ or $i = 2$. Moreover, the symmetric $(0, 2)$ -type tensor field g is a semi-Riemannian metric because $\det G = \frac{1}{16} \neq 0$. Therefore g is non degenerate, and

$$\det(G - \lambda I) = -\left(\frac{1}{2} + \lambda\right)^2 \left(\frac{1}{2} - \lambda\right) (\lambda - 1) (\lambda^2 - \left(\frac{3}{2} + 2(y^1)^2 + 2(y^2)^2\right) \lambda + \frac{1}{2}),$$

so, since the signs of eigenvalues are independent from the coordinates, the index of g is constant. We note that in this example ξ_1 and ξ_2 are spacelike. One proves that M is an indefinite \mathcal{S} -manifold.

Example 4.3 The third example is $M = (\mathbb{R}_1^4, \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ constructed as follows. Denoting the standard coordinates with $\{x, y, z^1, z^2\}$, we endow \mathbb{R}^4 with the structure $(\varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ where

$$\xi_\alpha = \frac{\partial}{\partial z^\alpha}, \quad \eta^\alpha = dz^\alpha + y dx,$$

for any $\alpha \in \{1, 2\}$ and where the tensor fields φ and g are given by

$$F := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & y & 0 & 0 \end{pmatrix} \quad G := \begin{pmatrix} \frac{1}{2} & 0 & y & -y \\ 0 & \frac{1}{2} & 0 & 0 \\ y & 0 & 1 & 0 \\ -y & 0 & 0 & -1 \end{pmatrix}$$

respectively. An immediate computation shows that g is non-degenerate and its index is constant. In fact, we have $\det G = -\frac{1}{4}$, and

$$\det(G - \lambda I) = \left(\frac{1}{2} - \lambda\right) (\lambda^3 - \frac{1}{2} \lambda^2 - (2y^2 + 1) \lambda + \frac{1}{2}),$$

hence $\det G \neq 0$ and, using Cartesio's rule, we deduce that the index is 1. Therefore, the tensor field g is a Lorentzian metric. Now, we observe that ξ_1 is a spacelike vector field while ξ_2 is a timelike vector field. One can check that M is an indefinite \mathcal{S} -manifold.

5 Sectional curvature and φ -sectional curvature

In this section, we look for some results about the sectional curvature of indefinite \mathcal{S} -manifolds. Following the notations in ([15]), for the curvature tensor R we have $R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$, and $R(X, Y, Z, W) = g(R(Z, W, Y), X)$, for any $X, Y, Z, W \in \Gamma(TM)$.

A two-dimensional subspace π of the tangent space $T_p M$ is called *non-degenerate* if and only if we have $\Delta(\pi) = g_p(X, X)g_p(Y, Y) - g_p(X, Y)^2 \neq 0$ for any basis $\{X, Y\}$ of π . We know that if π is a non-degenerate 2-plane of $T_p M$ then we can define the *sectional curvature* $K_p(\pi)$ at p with respect to the 2-plane π , putting

$$K_p(\pi) = \frac{R_p(X, Y, X, Y)}{\Delta(\pi)} = \frac{g_p(R_p(X, Y, Y), X)}{\Delta(\pi)},$$

where $\pi = \text{span}\{X, Y\}$. In the following we denote $K_p(\pi) = K_p(X, Y)$.

Proposition 5.1 *In an indefinite \mathcal{S} -manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ one has:*

- a) *the distribution $\ker \varphi$ is integrable and flat;*
- b) *the sectional curvatures $K(X, \xi_\alpha) = \varepsilon_\alpha$, for any $\alpha \in \{1, \dots, r\}$, and non lightlike $X \in \text{Im } \varphi$.*

Proof. For $X, Y \in \ker \varphi$ we have $X = f^\alpha \xi_\alpha$, $Y = t^\beta \xi_\beta$ then $[X, Y] = [f^\alpha \xi_\alpha, t^\beta \xi_\beta] = f^\alpha \xi_\alpha(t^\beta) \xi_\beta - t^\beta \xi_\beta(f^\alpha) \xi_\alpha \in \ker \varphi$ and $\ker \varphi$ is integrable. Furthermore, since $\nabla_{\xi_\alpha} \xi_\beta = 0$ and $[\xi_\alpha, \xi_\beta] = 0$, we have $R(\xi_\alpha, \xi_\beta, \xi_\gamma) = 0$ and $\ker \varphi$ is flat. Note that a) holds also for indefinite almost \mathcal{S} -manifolds. Now, being M an indefinite \mathcal{S} -manifold, we know that $\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi X$, $\mathcal{L}_{\xi_\alpha} \varphi = 0$ and we have

$$R(\xi_\alpha, X, \xi_\beta) = -\varepsilon_\beta \nabla_{\xi_\alpha}(\varphi X) + \varepsilon_\beta \varphi[\xi_\alpha, X] = \varepsilon_\beta(\varphi[\xi_\alpha, X] - [\xi_\alpha, \varphi X] - \nabla_{\varphi X} \xi_\alpha) = \varepsilon_\beta \varepsilon_\alpha \varphi^2 X.$$

So, for $X \in \text{Im } \varphi$, X non lightlike, we have $K(X, \xi_\alpha) = -\frac{\varepsilon_\alpha g(\varphi^2 X, X)}{g(X, X)} = \varepsilon_\alpha$. □

As usual, we say that a 2-plane π in $T_p M$, $p \in M$, is a φ -plane if $\pi = \text{span}\{X, \varphi X\}$ with $X \in \mathfrak{D}_p$, and the sectional curvature at p of such a plane, with X a non lightlike vector, is said the φ -sectional curvature at p and is denoted by $H_p(X)$.

We shall prove that on an indefinite \mathcal{S} -manifold, as in the Sasakian case, the φ -sectional curvatures determine the sectional curvatures.

As in [3], we define a tensor field of type (0,4) given for any X, Y, Z, W in $\Gamma(TM)$ by

$$P(X, Y; Z, W) = \Phi(X, Z)g(Y, W) - \Phi(X, W)g(Y, Z) - \Phi(Y, Z)g(X, W) + \Phi(Y, W)g(X, Z).$$

The following lemmas can be easily proved.

Lemma 5.2 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite \mathcal{S} -manifold. Then:*

- a) $P(X, Y; Z, W) = -P(Z, W; X, Y)$, for any $X, Y, Z, W \in \Gamma(TM)$,
- b) $P(X, Y; X, \varphi Y) = g(X, \varphi Y)^2 + g(X, Y)^2 - \varepsilon_X \varepsilon_Y$, where X, Y are unit vector fields in \mathfrak{D} and $\varepsilon_X = g(X, X)$ and $\varepsilon_Y = g(Y, Y)$.

Proposition 5.3 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite \mathcal{S} -manifold. Then, putting $\varepsilon = \sum_{\alpha=1}^r \varepsilon_\alpha$, for any $X, Y, Z, W \in \Gamma(TM)$*

$$g(R(X, Y, \varphi Z), W) + g(R(X, Y, Z), \varphi W) = -\varepsilon P(X, Y; Z, W) - Q(X, Y; Z, W)$$

where

$$\begin{aligned} Q(X, Y; Z, W) = & g(W, \varphi Y)(\varepsilon(g(X, Z) - g(\varphi X, \varphi Z)) - \bar{\eta}(Z)\bar{\eta}(X)) \\ & - g(W, \varphi X)(\varepsilon(g(Y, Z) - g(\varphi Y, \varphi Z)) - \bar{\eta}(Z)\bar{\eta}(Y)) \\ & - g(Z, \varphi Y)(\varepsilon(g(X, W) - g(\varphi X, \varphi W)) - \bar{\eta}(X)\bar{\eta}(W)) \\ & + g(Z, \varphi X)(\varepsilon(g(Y, W) - g(\varphi Y, \varphi W)) - \bar{\eta}(Y)\bar{\eta}(W)). \end{aligned}$$

Moreover if $X, Y, Z, W \in \mathfrak{D}$ then obviously $Q(X, Y; Z, W) = 0$ and the following statements hold:

- a) $g(R(\varphi X, \varphi Y, \varphi Z), \varphi W) = g(R(X, Y, Z), W);$
- b) $g(R(X, \varphi X, Y), \varphi Y) = g(R(X, Y, X), Y) + g(R(X, \varphi Y, X), \varphi Y) - 2\varepsilon P(X, Y, X, \varphi Y);$
- c) $g(R(\varphi X, Y, \varphi X), Y) = g(R(X, \varphi Y, X), \varphi Y).$

Remark 5.4 We remark that ε can vanish only if r is an even number and the number of timelike characteristic vector fields is equal to the number of spacelike characteristic vector fields. Moreover, $\varepsilon = 0$ means that $g(\bar{\xi}, \bar{\xi}) = 0$, i.e. $\bar{\xi} = \sum_{\alpha=1}^r \xi_\alpha$ is a lightlike vector field.

We put

$$B(X, Y) = g(R(X, Y, X), Y), \quad X, Y \in \Gamma(TM)$$

and

$$D(X) = B(X, \varphi X), \quad X \in \Gamma(\mathfrak{D}).$$

The following Lemma, of which we omit the long proof, gives the useful expression of $B(X, Y)$, for any $X, Y \in \Gamma(\mathfrak{D})$.

Lemma 5.5 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite \mathcal{S} -manifold. Then, for any $X, Y \in \Gamma(\mathfrak{D})$,*

$$\begin{aligned} B(X, Y) = \frac{1}{32} \{ & 3D(X + \varphi Y) + 3D(X - \varphi Y) - D(X + Y) \\ & - D(X - Y) - 4D(X) - 4D(Y) + 24\varepsilon P(X, Y; X, \varphi Y) \}. \end{aligned} \quad (8)$$

Using the previous Lemmas it is possible to compute the sectional curvature of a non degenerate 2-plane $\pi = \text{span}\{X, Y\}$ of \mathfrak{D}_p , as follows.

Proposition 5.6 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite \mathcal{S} -manifold and p in M . We consider a non degenerate 2-plane $\pi = \text{span}\{X, Y\}$ of \mathfrak{D}_p , where X and Y are unit vectors of \mathfrak{D}_p . Then the sectional curvature $K_p(X, Y)$ is given by*

$$\begin{aligned} K_p(X, Y) = \frac{1}{32(\varepsilon_X \varepsilon_Y - g(X, Y)^2)} \{ & 3(\varepsilon_X + \varepsilon_Y + 2g(X, \varphi Y))^2 H_p(X + \varphi Y) \\ & + 3(\varepsilon_X + \varepsilon_Y - 2g(X, \varphi Y))^2 H_p(X - \varphi Y) - (\varepsilon_X + \varepsilon_Y + 2g(X, Y))^2 H_p(X + Y) \\ & - (\varepsilon_X + \varepsilon_Y - 2g(X, Y))^2 H_p(X - Y) - 4H_p(X) - 4H_p(Y) \\ & + 24\varepsilon(g(X, \varphi Y)^2 + g(X, Y)^2 - \varepsilon_X \varepsilon_Y) \}. \end{aligned}$$

Proof. We note that if $X \in \mathfrak{D}_p$ we have $D_p(X) = B_p(X, \varphi X) = g_p(R_p(X, \varphi X, X), \varphi X) = -g_p(X, X)^2 H_p(X)$ and if X and Y are unit vectors of \mathfrak{D}_p , we find

$$g(X + \varphi Y, X + \varphi Y) = \varepsilon_X + \varepsilon_Y + 2g(X, \varphi Y), \quad g(X + Y, X + Y) = \varepsilon_X + \varepsilon_Y + 2g(X, Y).$$

Being $\Delta(\pi) = \varepsilon_X \varepsilon_Y - g_p(X, Y)^2$, we get $K_p(\pi) = -g_p(R_p(X, Y, X), Y) / \Delta(\pi) = -B_p(X, Y) / \Delta(\pi)$. Then, using (8) and Lemma 5.2, we get the required formula. \square

Remark 5.7 We note that if $X \in \Gamma(\mathfrak{D})$ is a unit vector field we have

$$R(\xi_\alpha, X, \xi_\beta) = -\varepsilon_\beta \varepsilon_\alpha X, \quad R(X, \xi_\alpha, X) = -\varepsilon_X \varepsilon_\alpha \bar{\xi}.$$

In fact, if $Y \in \Gamma(TM)$, for any $\alpha \in \{1, \dots, r\}$, we have

$$\begin{aligned} g(R(X, \xi_\alpha, X), Y) &= -g(R(X, Y, \xi_\alpha), X) = \varepsilon_\alpha g(\nabla_X(\varphi Y) - \nabla_Y(\varphi X) - \varphi[X, Y], X) \\ &= \varepsilon_\alpha g((\nabla_X \varphi)Y - (\nabla_Y \varphi)X, X) = \varepsilon_\alpha g(-\bar{\eta}(Y)X - \bar{\eta}(X)\varphi^2 Y, X) \\ &= -\varepsilon_X \varepsilon_\alpha \bar{\eta}(Y) = -\varepsilon_X \varepsilon_\alpha g(\bar{\xi}, Y). \end{aligned}$$

Finally, if $X, Y \in \Gamma(\mathfrak{D})$ and $Z \in \Gamma(TM)$ then we get

$$g(R(X, \xi_\alpha, Y), Z) = -\varepsilon_\alpha g(Y, X) \bar{\eta}(Z) = -\varepsilon_\alpha g(Y, X) g(\bar{\xi}, Z).$$

Theorem 5.8 *The φ -sectional curvatures completely determine the sectional curvatures of an indefinite \mathcal{S} -manifold.*

Proof. We show that for any $p \in M$ and for any non degenerate 2-plane $\pi = \text{span}\{X, Y\}$ in $T_p(M)$ the sectional curvature $K_p(X, Y)$ is uniquely determined by the φ -sectional curvature. In the sequel of the proof we suppose that $p \in M$ is fixed. If $X, Y \in \mathfrak{D}_p$, then we apply the previous Proposition and if X or Y is ξ_α , for any $\alpha \in \{1, \dots, r\}$, we have already seen that $K_p(X, Y) = \varepsilon_\alpha$. If $X, Y \in T_p M$, they can be written in the following way:

$$X = aZ + \eta^\alpha(X)\xi_\alpha, \quad Y = bW + \eta^\alpha(Y)\xi_\alpha,$$

where $Z, W \in \mathfrak{D}$, $g_p(Z, Z) = \varepsilon_Z$, $g_p(W, W) = \varepsilon_W$, and a and b must satisfy:

$$a^2 \varepsilon_Z = \varepsilon_X - \varepsilon_\alpha (\eta^\alpha(X))^2, \quad b^2 \varepsilon_W = \varepsilon_Y - \varepsilon_\alpha (\eta^\alpha(Y))^2.$$

Therefore, we compute

$$\begin{aligned} g_p(R_p(X, Y, X), Y) &= a^2 b^2 g_p(R_p(Z, W, Z), W) + 2a^2 b \eta^\beta(Y) g_p(R_p(Z, W, Z), \xi_\beta) \\ &\quad + 2ab^2 \eta^\alpha(X) g_p(R_p(Z, W, \xi_\alpha), W) + 2ab \eta^\alpha(X) \eta^\beta(Y) g_p(R_p(Z, W, \xi_\alpha), \xi_\beta) \\ &\quad + a^2 \eta^\beta(Y) \eta^\delta(Y) g_p(R_p(Z, \xi_\beta, Z), \xi_\delta) + 2ab \eta^\beta(Y) \eta^\alpha(X) g_p(R_p(Z, \xi_\beta, \xi_\alpha), W) \\ &\quad + 2a \eta^\beta(Y) \eta^\alpha(X) \eta^\delta(Y) g_p(R_p(Z, \xi_\beta, \xi_\alpha), \xi_\delta) + b^2 \eta^\alpha(X) \eta^\gamma(X) g_p(R_p(\xi_\alpha, W, \xi_\gamma), W) \\ &\quad + 2b \eta^\alpha(X) \eta^\beta(Y) \eta^\gamma(X) g_p(R_p(\xi_\alpha, Z, \xi_\gamma), \xi_\beta) + \eta^\alpha(X) \eta^\beta(Y) \eta^\gamma(X) \eta^\delta(Y) g_p(R_p(\xi_\alpha, \xi_\beta, \xi_\gamma), \xi_\delta). \end{aligned} \quad (9)$$

Now, separately we take the terms of previous expression into account, using Remark 5.7 and the Bianchi identity, as follows:

$$\begin{aligned} g_p(R_p(Z, W, Z), \xi_\beta) &= g_p(R_p(Z, \xi_\beta, Z), W) = -\varepsilon_Z \varepsilon_\beta g_p(\bar{\xi}, W) = 0, \\ g_p(R_p(Z, W, \xi_\alpha), W) &= g_p(R_p(\xi_\alpha, W, Z), W) = g_p(R_p(W, \xi_\alpha, W), Z) = -\varepsilon_W \varepsilon_\alpha g_p(\bar{\xi}, Z) = 0, \\ g_p(R_p(Z, W, \xi_\alpha), \xi_\beta) &= -g_p(R_p(Z, \xi_\alpha, \xi_\beta), W) - g_p(R_p(Z, \xi_\beta), \xi_\alpha, W) = g_p(R_p(\xi_\alpha, Z, \xi_\beta), W) \\ &\quad + \varepsilon_\beta g_p(Z, W) g_p(\bar{\xi}, \xi_\alpha) = -\varepsilon_\beta \varepsilon_\alpha g_p(Z, W) + \varepsilon_\beta \varepsilon_\alpha g_p(Z, W) = 0, \\ g_p(R_p(Z, \xi_\beta, \xi_\alpha), W) &= -g_p(R_p(Z, \xi_\beta, W) \xi_\alpha) = \varepsilon_\beta g_p(Z, W) g_p(\bar{\xi}, \xi_\alpha) = \varepsilon_\beta \varepsilon_\alpha g_p(Z, W), \\ g_p(R_p(Z, \xi_\beta, \xi_\alpha), \xi_\delta) &= -g_p(R_p(\xi_\beta, Z, \xi_\alpha), \xi_\delta) = \varepsilon_\beta \varepsilon_\alpha g_p(Z, \xi_\delta) = 0, \\ g_p(R_p(\xi_\alpha, W, \xi_\gamma), \xi_\beta) &= \varepsilon_\gamma \varepsilon_\alpha g_p(Z, \xi_\beta) = 0. \end{aligned}$$

Therefore, replacing the previous expressions in (9), we have:

$$\begin{aligned} g_p(R_p(X, Y, X), Y) &= a^2 b^2 g_p(R_p(Z, W, Z), W) - a^2 \varepsilon_Z \bar{\eta}(Y) \bar{\eta}(Y) \\ &\quad + 2ab \bar{\eta}(Y) \bar{\eta}(X) g_p(Z, W) - b^2 \varepsilon_W \bar{\eta}(X) \bar{\eta}(X). \end{aligned}$$

Hence, being $K_p(X, Y) = -\varepsilon_X \varepsilon_Y g_p(R_p(X, Y, X), Y)$, we deduce

$$\begin{aligned} K_p(X, Y) &= \varepsilon_X \varepsilon_Y \{a^2 b^2 g_p(R_p(Z, W, W), Z) - 2ab \bar{\eta}(Y) \bar{\eta}(X) g_p(Z, W) \\ &\quad + b^2 \varepsilon_W \bar{\eta}(X)^2 + a^2 \varepsilon_Z \bar{\eta}(Y)^2\}. \end{aligned} \quad (10)$$

Now, we note that

$$g_p(Z, W) = \frac{1}{ab} g_p(X - \eta^\alpha(X)\xi_\alpha, Y - \eta^\beta(Y)\xi_\beta) + \eta^\alpha(X) \eta^\beta(Y) g_p(\xi_\alpha, \xi_\beta) = -\frac{1}{ab} \varepsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y),$$

$$\begin{aligned}
g_p(R_p(Z, W, W), Z) &= [\epsilon_Z \epsilon_W - g_p(Z, W)^2] K_p(Z, W) \\
&= \frac{1}{a^2 b^2} [a^2 \epsilon_Z b^2 \epsilon_W - (\epsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y))^2] K_p(Z, W) \\
&= \frac{1}{a^2 b^2} [(\epsilon_X - \epsilon_\alpha \eta^\alpha(X))^2 (\epsilon_Y - \epsilon_\alpha \eta^\alpha(Y))^2 \\
&\quad - (\epsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y))^2] K_p(Z, W).
\end{aligned}$$

Thus, (10) becomes

$$\begin{aligned}
K_p(X, Y) &= \epsilon_X \epsilon_Y \{ [(\epsilon_X - \epsilon_\alpha \eta^\alpha(X))^2 (\epsilon_Y - \epsilon_\beta \eta^\beta(Y))^2 \\
&\quad - (\epsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y))^2] K_p(Z, W) + 2\bar{\eta}(Y)\bar{\eta}(X) \epsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y) \\
&\quad + (\epsilon_Y - \epsilon_\beta \eta^\beta(Y))^2 \bar{\eta}(X)^2 + (\epsilon_X - \epsilon_\alpha \eta^\alpha(X))^2 \bar{\eta}(Y)^2 \},
\end{aligned}$$

and this completes the proof, since $K_p(Z, W)$ is given as in Proposition 5.6. \square

We recall the following result.

Lemma 5.9 ([16]) *Let (V, g) be a semi-Euclidean vector space and R a $(0, 4)$ -type tensor on V such that for any $X, Y, Z, W \in V$ the following conditions hold:*

- a) $R(X, Y, Z, W) = -R(Y, X, Z, W)$,
- b) $R(X, Y, Z, W) = -R(X, Y, W, Z)$,
- c) $R(X, Y, Z, W) = R(Z, W, X, Y)$,
- d) $\mathfrak{S}_{Y, Z, W} R(X, Y, Z, W) = 0$.

If $R(X, Y, X, Y) = 0$ for any linearly independent and non lightlike vectors $X, Y \in V$, then $R = 0$. Moreover, if R and S are $(0, 4)$ -type tensors on V such that the conditions (a-d) are satisfied and $R(X, Y, X, Y) = S(X, Y, X, Y)$ for any $X, Y \in V$ linearly independent non lightlike vectors, then $R = S$.

Proposition 5.10 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite \mathcal{S} -manifold, T and S be $(0, 4)$ -type tensor fields on M such that the following conditions hold:*

- i) $T(X, Y, Z, W) = -T(Y, X, Z, W)$, $S(X, Y, Z, W) = -S(Y, X, Z, W)$, $X, Y, Z, W \in \Gamma(TM)$
- ii) $T(X, Y, Z, W) = -T(X, Y, W, Z)$, $S(X, Y, Z, W) = -S(X, Y, W, Z)$, $X, Y, Z, W \in \Gamma(TM)$
- iii) $T(X, Y, Z, W) = T(Z, W, X, Y)$, $S(X, Y, Z, W) = S(Z, W, X, Y)$, $X, Y, Z, W \in \Gamma(TM)$
- iv) $\mathfrak{S}_{Y, Z, W} T(X, Y, Z, W) = 0$, $\mathfrak{S}_{Y, Z, W} S(X, Y, Z, W) = 0$, $X, Y, Z, W \in \Gamma(TM)$
- v) for any $X, Y, Z, W \in \Gamma(\mathfrak{D})$

$$\begin{aligned}
T(X, Y, \varphi Z, W) + T(X, Y, Z, \varphi W) &= \epsilon P(X, Y; Z, W) \\
S(X, Y, \varphi Z, W) + S(X, Y, Z, \varphi W) &= \epsilon P(X, Y; Z, W)
\end{aligned}$$

- vi) for any $X, Y \in \Gamma(\mathfrak{D})$ and for any $\alpha, \beta, \gamma, \delta \in \{1, \dots, r\}$

- (a) $T(X, \xi_\alpha, X, Y) = S(X, \xi_\alpha, X, Y)$,
- (b) $T(\xi_\alpha, X, \xi_\beta, Y) = S(\xi_\alpha, X, \xi_\beta, Y)$,
- (c) $T(\xi_\alpha, X, \xi_\beta, \xi_\gamma) = S(\xi_\alpha, X, \xi_\beta, \xi_\gamma)$,

$$(d) \quad T(\xi_\alpha, \xi_\beta, \xi_\gamma, \xi_\delta) = S(\xi_\alpha, \xi_\beta, \xi_\gamma, \xi_\delta) .$$

Then, if $T(X, \varphi X, X, \varphi X) = S(X, \varphi X, X, \varphi X)$ for any $X \in \Gamma(\mathfrak{D})$ non lightlike vector field, one has $T = S$.

Proof. It is to verify that $v)$ implies that for any X', Y', Z', W' in $\Gamma(\mathfrak{D})$

$$T(\varphi X', \varphi Y', \varphi Z', \varphi W') = T(X', Y', Z', W'),$$

and, using the above formula, we obtain $T(\varphi X', \varphi Y', Z', W') = T(X', Y', \varphi Z', \varphi W')$. Analogously, for the tensor field S we have $S(\varphi X', \varphi Y', Z', W') = S(X', Y', \varphi Z', \varphi W')$.

Now, being φ_p an almost complex structure on \mathfrak{D}_p for any $p \in M$, from a well-known result analogous to the Lemma 5.9 ([1]), in the case of a real vector space endowed with an almost complex structure, we deduce $T(X', Y', Z', W') = S(X', Y', Z', W')$. Then, in particular, we have $T(X', Y', X', Y') = S(X', Y', X', Y')$.

Now, if $X, Y \in \Gamma(TM)$ are linearly independent and non lightlike, we compute $T(X, Y, X, Y)$ and $S(X, Y, X, Y)$, writing $X = X' + \eta^\alpha(X)\xi_\alpha$ and $Y = Y' + \eta^\alpha(Y)\xi_\alpha$, and likewise to (9), by the $\mathfrak{F}(M)$ -linearity of T and S , using $vi)$, we get $T(X, Y, X, Y) = S(X, Y, X, Y)$. \square

Remark 5.11 Using Remark 5.7 and Proposition 5.1, the Riemannian (0,4)-type curvature tensor field R satisfies the properties listen in Proposition 5.10. Thus, it is uniquely determined by the φ -sectional curvature.

Theorem 5.12 *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite \mathcal{S} -manifold. Then the φ -sectional curvature c is pointwise constant, $c \in \mathfrak{F}(M)$, if and only if the Riemannian (0,4)-type curvature tensor field R is given by*

$$\begin{aligned} R(X, Y, Z, W) = & -\frac{c+3\varepsilon}{4}\{g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)\} \\ & -\frac{c-\varepsilon}{4}\{\Phi(W, X)\Phi(Z, Y) - \Phi(Z, X)\Phi(W, Y) + 2\Phi(X, Y)\Phi(W, Z)\} \\ & -\{\bar{\eta}(W)\bar{\eta}(X)g(\varphi Z, \varphi Y) - \bar{\eta}(W)\bar{\eta}(Y)g(\varphi Z, \varphi X) + \bar{\eta}(Y)\bar{\eta}(Z)g(\varphi W, \varphi X) \\ & -\bar{\eta}(Z)\bar{\eta}(X)g(\varphi W, \varphi Y)\}. \end{aligned} \quad (11)$$

Proof. We suppose that the φ -sectional curvature c is pointwise constant and in order to prove (11), denote by $S(X, Y, Z, W)$ the right-hand side of (11). Obviously S is a tensor field of type (0,4) on M , and we shall prove that S coincides with R . To this end it is easy to check that for any $X, Y, Z, W \in \Gamma(TM)$ we have the properties of skew-symmetry $-S(X, Y, W, Z) = S(X, Y, Z, W) = -S(Y, X, Z, W)$ and the Bianchi identity $\mathfrak{S}_{Y,Z,W}S(X, Y, Z, W) = 0$, while the property *iii)* of Proposition 5.10, $S(X, Y, Z, W) = S(Z, W, X, Y)$, follows by the Bianchi identity and the skew-symmetries.

Now, for $X, Y, Z, W \in \Gamma(\mathfrak{D})$, computing $S(X, Y, Z, \varphi W) + S(X, Y, \varphi Z, W)$ we get

$$\begin{aligned} S(X, Y, Z, \varphi W) + S(X, Y, \varphi Z, W) = & -\frac{c}{4}\{g(Y, Z)\Phi(X, W) - g(X, Z)\Phi(Y, W) + \Phi(Y, Z)g(X, W) \\ & - \Phi(X, Z)g(Y, W) + g(W, X)\Phi(Z, Y) - \Phi(Z, X)g(W, Y) + \Phi(W, X)g(Z, Y) - g(Z, X)\Phi(W, Y)\} \\ & -\frac{\varepsilon}{4}\{3\Phi(X, W)g(Z, Y) - 3\Phi(Y, W)g(X, Z) + 3g(X, W)\Phi(Y, Z) - 3g(Y, W)\Phi(X, Z) \\ & + \Phi(Y, Z)g(W, X) - \Phi(X, Z)g(W, Y) + \Phi(X, W)g(Z, Y) - \Phi(Y, W)g(Z, X)\} \\ = & -\varepsilon\{\Phi(X, W)g(Z, Y) - \Phi(X, Z)g(Y, W) - \Phi(Y, W)g(X, Z) + g(X, W)\Phi(Y, Z)\} \\ = & \varepsilon P(X, Y; Z, W). \end{aligned}$$

We continue verifying *vi*) of Proposition 5.10, and obtaining $S(X, \xi_\alpha, X, Y) = 0 = R(X, \xi_\alpha, X, Y)$, $S(\xi_\alpha, X, \xi_\beta, \xi_\gamma) = 0 = R(\xi_\delta, X, \xi_\beta, \xi_\gamma)$, $S(\xi_\alpha, \xi_\delta, \xi_\beta, \xi_\gamma) = 0 = R(\xi_\delta, \xi_\delta, \xi_\beta, \xi_\gamma)$ and

$$\begin{aligned} S(\xi_\alpha, X, \xi_\beta, Y) &= -\frac{c+3\varepsilon}{4}\{g(\varphi X, \varphi \xi_\beta)g(\varphi \xi_\alpha, \varphi Y) - g(\varphi \xi_\alpha, \varphi \xi_\beta)g(\varphi X, \varphi Y)\} \\ &\quad - \frac{c-\varepsilon}{4}\{\Phi(Y, \xi_\alpha)\Phi(\xi_\beta, X) - \Phi(\xi_\beta, \xi_\alpha)\Phi(Y, X) + 2\Phi(\xi_\alpha, X)\Phi(Y, \xi_\beta)\} \\ &\quad - \{\bar{\eta}(Y)\bar{\eta}(\xi_\alpha)g(\varphi \xi_\beta, \varphi X) - \bar{\eta}(Y)\bar{\eta}(X)g(\varphi \xi_\beta, \varphi \xi_\alpha) + \bar{\eta}(X)\bar{\eta}(\xi_\beta)g(\varphi Y, \varphi \xi_\alpha) \\ &\quad - \bar{\eta}(\xi_\beta)\bar{\eta}(\xi_\alpha)g(\varphi Y, \varphi X)\} = \varepsilon_\alpha \varepsilon_\beta g(X, Y) = R(\xi_\alpha, X, \xi_\beta, Y). \end{aligned}$$

For any $X \in \Gamma(\mathfrak{D})$ non lightlike vector field, we compute $S(X, \varphi X, X, \varphi X)$, obtaining:

$$\begin{aligned} S(X, \varphi X, X, \varphi X) &= -\frac{c+3\varepsilon}{4}\{g(\varphi^2 X, \varphi X)g(\varphi X, \varphi^2 X) - g(\varphi X, \varphi X)g(\varphi^2 X, \varphi^2 X)\} \\ &\quad - \frac{c-\varepsilon}{4}\{\Phi(\varphi X, X)\Phi(X, \varphi X) - \Phi(X, X)\Phi(\varphi X, \varphi X) + 2\Phi(X, \varphi X)\Phi(\varphi X, X)\} \\ &\quad - \{\bar{\eta}(\varphi X)\bar{\eta}(X)g(\varphi X, \varphi^2 X) - \bar{\eta}(\varphi X)\bar{\eta}(\varphi X)g(\varphi X, \varphi X) \\ &\quad + \bar{\eta}(\varphi X)\bar{\eta}(X)g(\varphi^2 X, \varphi X) - \bar{\eta}(X)\bar{\eta}(X)g(\varphi^2 X, \varphi^2 X)\} \\ &= \frac{c+3\varepsilon}{4}g(X, X)^2 - \frac{c-\varepsilon}{4}\{-g(X, X)^2 - 2g(X, X)^2\} \\ &= \frac{c+3\varepsilon}{4}g(X, X)^2 + 3\frac{c-\varepsilon}{4}g(X, X)^2 = cg(X, X)^2. \end{aligned} \tag{12}$$

Moreover, since by definition of φ -sectional curvature we have

$$R(X, \varphi X, X, \varphi X) = cg(X, X)^2. \tag{13}$$

from (12) and (13) we get $R(X, \varphi X, X, \varphi X) = S(X, \varphi X, X, \varphi X)$, and, using Proposition 5.10, the previous Remark and the properties of the tensor field S , we obtain $R(X, Y, Z, W) = S(X, Y, Z, W)$, for any $X, Y, Z, W \in \Gamma(TM)$, that is the formula (11).

Conversely, if we assume (11), choosing a point $p \in M$ and a φ -plane $\pi = \text{span}\{X, \varphi X\}$, with $X \in \mathfrak{D}_p$ non lightlike vector, by direct computation, omitting the point p , we have

$$H(X) = \frac{c+3\varepsilon}{4g(X, X)^2}g(X, X)^2 + 3\frac{c-\varepsilon}{4g(X, X)^2}g(X, X)^2 = c.$$

□

6 Sectional Curvature in the case $\varepsilon = 0$, an example

In this section we consider the case $\varepsilon = 0$, as already pointed out, $r = 2p$ and ξ_1, \dots, ξ_p are timelike vector field, $\xi_{p+1}, \dots, \xi_{2p}$ are spacelike vector field. We call such a manifold a *special indefinite \mathcal{S} -manifold*. Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a special indefinite \mathcal{S} -manifold. The tensor Q is given by

$$\begin{aligned} Q(X, Y; Z, W) &= -g(W, \varphi Y)\bar{\eta}(Z)\bar{\eta}(X) + g(W, \varphi X)\bar{\eta}(Z)\bar{\eta}(Y) + g(Z, \varphi Y)\bar{\eta}(X)\bar{\eta}(W) \\ &\quad - g(Z, \varphi X)\bar{\eta}(Y)\bar{\eta}(W), \end{aligned}$$

and

$$g(R(X, Y, \varphi Z), W) + g(R(X, Y, Z), \varphi W) = -Q(X, Y; Z, W)$$

Moreover, being $Q(X, Y; Z, W) = 0$ for any $X, Y, Z, W \in \mathfrak{D}$, we have

- a) $g(R(\varphi X, \varphi Y, \varphi Z), \varphi W) = g(R(X, Y, Z), W)$;
- b) $g(R(X, \varphi X, Y), \varphi Y) = g(R(X, Y, X), Y) + g(R(X, \varphi Y, X), \varphi Y)$;
- c) $g(R(\varphi X, Y, \varphi X), Y) = g(R(X, \varphi Y, X), \varphi Y)$.

Furthermore, for $X, Y \in \Gamma(\mathfrak{D})$

$$B(X, Y) = \frac{1}{32} \{3D(X + \varphi Y) + 3D(X - \varphi Y) - D(X + Y) - D(X - Y) - 4D(X) - 4D(Y)\},$$

and for a non degenerate 2-plane $\pi = \text{span}\{X, Y\}$ of \mathfrak{D}_p , where X and Y are unit vectors of \mathfrak{D}_p ,

$$\begin{aligned} K_p(X, Y) = & \frac{1}{32(\varepsilon_X \varepsilon_Y - g(X, Y)^2)} \{3(\varepsilon_X + \varepsilon_Y + 2g(X, \varphi Y))^2 H_p(X + \varphi Y) \\ & + 3(\varepsilon_X + \varepsilon_Y - 2g(X, \varphi Y))^2 H_p(X - \varphi Y) - (\varepsilon_X + \varepsilon_Y + 2g(X, Y))^2 H_p(X + Y) \\ & - (\varepsilon_X + \varepsilon_Y - 2g(X, Y))^2 H_p(X - Y) - 4H_p(X) - 4H_p(Y)\}. \end{aligned}$$

Finally we have that the φ -sectional curvature c is pointwise constant, $c \in \mathfrak{F}(M)$, if and only if the Riemannian (0,4)-type curvature tensor field R is given by

$$\begin{aligned} R(X, Y, Z, W) = & -\frac{c}{4} \{g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W) \\ & + \Phi(W, X)\Phi(Z, Y) - \Phi(Z, X)\Phi(W, Y) + 2\Phi(X, Y)\Phi(W, Z)\} \\ & - \{\bar{\eta}(W)\bar{\eta}(X)g(\varphi Z, \varphi Y) - \bar{\eta}(W)\bar{\eta}(Y)g(\varphi Z, \varphi X) \\ & + \bar{\eta}(Y)\bar{\eta}(Z)g(\varphi W, \varphi X) - \bar{\eta}(Z)\bar{\eta}(X)g(\varphi W, \varphi Y)\}. \end{aligned} \quad (14)$$

An example of a special indefinite \mathcal{S} -manifold is $M = (\mathbb{R}_1^4, \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$, which is described in Example 4.3. We observe that the metric is Lorentzian, ξ_1 is a spacelike vector field while ξ_2 is a timelike vector field, then, since $\varepsilon = 0$, the structure is a special indefinite \mathcal{S} -structure. Now, we compute the tensor field Q on some relevant set of vector fields, the sectional curvature and φ -sectional curvature. We know that $Q = 0$ on \mathfrak{D} , moreover we have

$$\begin{aligned} Q(\xi_1, Y; Z, W) &= -Q(\xi_2, Y; Z, W) = -g(W, \varphi Y)\bar{\eta}(Z) + g(Z, \varphi Y)\bar{\eta}(W) = 0, \\ Q(\xi_\alpha, Y; \xi_\beta, W) &= Q(Y, \xi_\alpha; W, \xi_\beta) = -\varepsilon_\alpha \varepsilon_\beta g(W, \varphi Y), \end{aligned} \quad (15)$$

for any $Y, Z, W \in \Gamma(\mathfrak{D})$ and for any $\alpha, \beta \in \{1, 2\}$. Equation (15) shows that Q never vanishes. Now, computing the Christoffel's symbols we obtain:

$$\Gamma_{12}^3 = \Gamma_{12}^4 = \frac{1}{2}, \quad \Gamma_{13}^2 = -\Gamma_{14}^2 = -\Gamma_{23}^1 = \Gamma_{24}^1 = -1, \quad \Gamma_{23}^3 = \Gamma_{23}^4 = -\Gamma_{24}^3 = -\Gamma_{24}^4 = -y,$$

whereas the other Γ_{ij}^k vanish. To compute the φ -sectional curvature, being \mathfrak{D} globally spanned by $X = \frac{\partial}{\partial x} - y\xi_1 - y\xi_2$ and $Y = \varphi X = \frac{\partial}{\partial y}$, we value $H(X)$. So, we have

$$\begin{aligned} R(X, \varphi X, X) &= \nabla_X \left(\Gamma_{21}^h - y(\Gamma_{23}^h + \Gamma_{24}^h) \frac{\partial}{\partial x^h} - \xi_1 - \xi_2 \right) - \nabla_{\xi_1} X - \nabla_{\xi_2} X \\ &= -\frac{1}{2} \nabla_X (\xi_1 + \xi_2) - (\Gamma_{31}^h - y(\Gamma_{33}^h + \Gamma_{34}^h) + \Gamma_{41}^h - y(\Gamma_{43}^h + \Gamma_{44}^h)) \frac{\partial}{\partial x^h} \\ &= [\Gamma_{11}^h - y(\Gamma_{31}^h + \Gamma_{41}^h) - y(\Gamma_{13}^h - y(\Gamma_{33}^h + \Gamma_{43}^h) + \Gamma_{14}^h - y(\Gamma_{34}^h + \Gamma_{44}^h))] \frac{\partial}{\partial x^h} = 0, \end{aligned}$$

$$g(X, X) = g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) - 2y\left(g\left(\frac{\partial}{\partial x}, \xi_1\right) + g\left(\frac{\partial}{\partial x}, \xi_2\right)\right) + y^2(g(\xi_1, \xi_1) + g(\xi_1, \xi_2) + g(\xi_2, \xi_2)) = \frac{1}{2}.$$

It follows that

$$H(X) = -\frac{1}{g(X, X)^2}g(R(X, \varphi X, X), \varphi X) = 0.$$

Then, M is an indefinite \mathcal{S} -space form with $c = 0 = \varepsilon$ and, from (14), the Riemannian curvature tensor field R is given by:

$$\begin{aligned} R(X, Y, Z, W) = & -\{\bar{\eta}(W)\bar{\eta}(X)g(\varphi Z, \varphi Y) - \bar{\eta}(W)\bar{\eta}(Y)g(\varphi Z, \varphi X) \\ & + \bar{\eta}(Y)\bar{\eta}(Z)g(\varphi W, \varphi X) - \bar{\eta}(Z)\bar{\eta}(X)g(\varphi W, \varphi Y)\}. \end{aligned}$$

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